A Reduced State Estimator for Orbital Heading Reference

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It is practical to estimate only yaw, using a single gyro, rather than estimating both roll and yaw with the more conventional two-gyro arrangement. In the development of observer theory, such a mechanization is termed a "reduced state estimator." A decoupled single-skewed-gyro gyrocompass is developed using estimation theory, and design curves are presented for parameter selection as a function of noise models for the gyro and horizon sensor. The modern development of this gyrocompass system is compared with earlier analyses showing the advantage of the modern approach. Noise models are discussed for each instrument and related to performance through the mechanization parameters for use in design. Design parameter selection is based on a numerical optimization technique. It is concluded that the reduced state estimator can provide reliable and accurate yaw measurements on the order of 0.1° without the cost and complexity of a stellar reference system or of the two-gyro gyrocompassing mechanism.

Nomenclature

b = constant gyro drift
c_i, d_i = polynomial coefficient numerator, denominator of Laplace transform
$E(\,)$ = expected value
J = cost function
n = orbital mean motion
Q_d = power spectral density of w
R = power spectral density of v
R_d = covariance of correlated gyro drift
s = Laplace transform operator
T_d = gyro drift correlation time
v = horizon sensor white noise
w = gyro white noise
α = gyro drift first order break frequency
β = orientation of gyro input axis, see Fig. 1.
γ = horizon sensor first-order break frequency
φ = orientation of satellite about gyro input axis, see Fig. 1
$\varphi_\perp$ = orientation of satellite about gyro output axis, see Fig. 1
φ_i = $i = 1, 2, 3$, Euler rotations about the body axes
ω = body angular velocity about the gyro input axis
$\omega_\perp$ = body angular velocity about the gyro output axis
ω_i = body angular velocity about the body axes
ω_d' = total gyro drift rate
ω_d = correlated gyro drift
$\dot{(\,)}$ = time derivative
$\hat{(\,)}$ = estimated quantities
$\tilde{(\,)}$ = error quantities $\triangleq \hat{(\,)} - (\,)$

Introduction

FOR modest required Earth-pointing accuracy, an Earth horizon sensor may be employed to measure pitch and roll motions of the satellite with respect to the local vertical. Accuracy is limited by poor definition of the Earth's horizon, even at infrared wavelengths, to several tenths of a degree at low altitudes. For synchronous communications satellites, horizon sensors can provide sensing to better than 0.1°.

Yaw motion about the local vertical produces no change in pitch and roll attitude, so horizon sensors alone cannot sense yaw attitude. An estimate of yaw angle can be made by a weighted integral of roll angle using knowledge of the kinematic coupling of yaw rate to roll angle and the roll rate-to yaw angle. The estimator so constructed is called an "orbital gyrocompass"; implied knowledge of roll and yaw rates means that the gyrocompass must include rate or rate integrating gyros to perform these measurements. Flight gyrocompassing systems[1] have employed a two-state estimator, with two gyros, to estimate both roll and yaw attitudes. This mechanization is of particular advantage when horizon sensor measurements are noisy and roll rate information is desired. A complete development of the two-state gyrocompass is given in Bowers et al.,[2] and more extensively using modern control theory to optimize the gains in Kortüm.[3]

If one is willing to accept some compromises in the noise rejection capabilities of the estimator, a single gyro may be employed to estimate only yaw.

Reduced State Gyrocompass

An estimator for satellite yaw attitude which requires only one additional state variable will now be described. The estimator is based on the kinematical equations of motion rather than those of the combined kinematics and dynamics. As a result, it will only produce estimates of yaw attitude. In previous studies,[3] modeling the dynamics to obtain rates has not proved very fruitful and will not be addressed here.

The orientation of the vehicle body axes in orbit is shown in Fig. 1. For small Euler rotations, φ_1, φ_2, φ_3, pitch motion is decoupled from roll and yaw, the order of Euler rotations is of no consequence, and the body rates, Euler angles, and orbit rate n are related by

$$\omega_1 = \dot{\varphi}_1 - n\varphi_2 \tag{1}$$

$$\omega_2 = \dot{\varphi}_2 + n\varphi_1 \tag{2}$$

if the yaw motion is to be observed. Roll attitude information is provided by an Earth horizon sensor and may include a significant amount of noise so that forming the derivative of this output may aggravate the noise contribution and make resolution of the gyrocompassing information impossible. This problem may be largely overcome by also measuring yaw-body rate which provides short-term information about yaw motion so that relatively high-frequency (compared to orbit rate) noise is no longer so significant. In a single-state estimator, a combination of both body rates may be detected by a single axis gyro with its input axis (IA) skewed in the roll-yaw plane as shown in Fig. 1.

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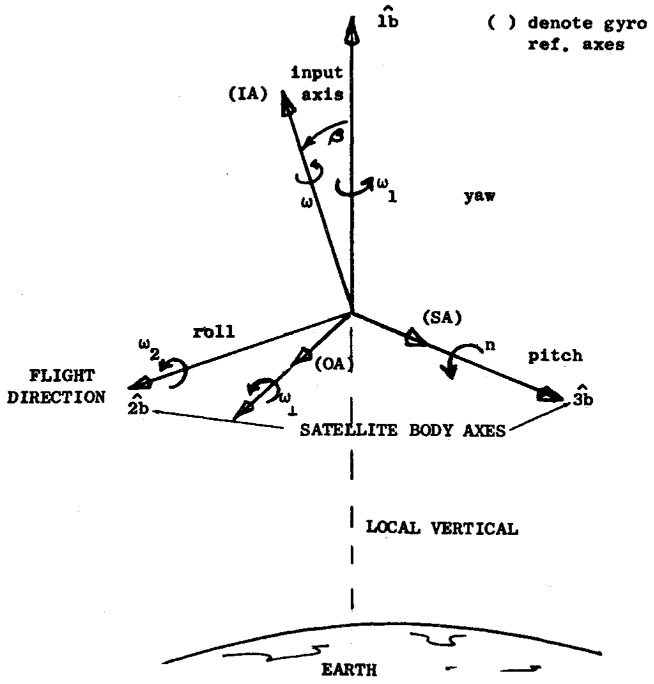

Fig. 1 Rigid body and gyrocompass reference frames.

The reference orbital rotation rate is about the $\hat{3}$b axis so the kinematics will be the same in all axes in the roll-yaw plane. Introducing two new angles φ and $\varphi_\perp$ as the small angle rotations about the gyro input axis (IA) and output axis (OA), and replacing φ_1 and φ_2 with them in Eq. (1) gives

$$\dot{\varphi} = n\varphi_\perp + \omega \tag{3}$$

The new angles are simply related to φ_1 and φ_2 by an Euler rotation about $\hat{3}$b through the angle β, viz.,

$$\varphi = \varphi_1 \cos\beta + \varphi_2 \sin\beta \tag{4}$$

$$\varphi_\perp = -\varphi_1 \sin\beta + \varphi_2 \cos\beta \tag{5}$$

Solving Eqs. (3–5) to eliminate $\varphi_\perp$ and φ_1

$$\dot{\varphi} = -n\tan\beta\varphi + (n\varphi_2/\cos\beta) + \omega \tag{6}$$

which provides an equation for the kinematics about the gyro input axes as a function of the roll angle which is measured with an horizon sensor. Define $\hat{\varphi}_2$ and $\hat{\omega}$ as the horizon sensor and gyro outputs, respectively, and v and $-\omega_d$ as their measurement errors.

The error $\tilde{\varphi} = \hat{\varphi} - \varphi$ is the principal variable in the estimator error equation which may be written as

$$\dot{\tilde{\varphi}} = -n\tan\beta\tilde{\varphi} + (nv/\cos\beta) - \omega_d \tag{7}$$

where the estimator for φ is exactly like Eq. (6) with φ, φ_2 and ω replaced with $\hat{\varphi}$, $\hat{\varphi}_2$ and $\hat{\omega}$. The error equation is the difference between the estimator equation and Eq. (6). To obtain yaw attitude, Eq. (4) is used to recover an estimate of φ_1 from the estimated value of $\hat{\varphi}$

$$\hat{\varphi}_1 = \hat{\varphi}/\cos\beta - \hat{\varphi}_2 \tan\beta \tag{8}$$

It is immaterial whether a rate gyro with an electronic-integrator or a rate-integrating gyro is used. The latter is shown in Fig. 2 which is a mechanization of the estimator and the solution for $\hat{\varphi}_1$ given in Eq. (8). It should be noted that the feedforward gains (characteristic of all reduced state estimators) make differentiation of the horizon sensor output unnecessary. This mechanization is identical to the one obtained by Bowers et al.[2] by heuristic arguments and is a form obtainable by using the reduced observer theory developed by Luenberger.[4] The latter requires a dynamical estimator for a new state variable which is a linear combination of the original states φ_1 and φ_2. This is automatically accomplished in the preceding development by writing the estimator equations in the skewed gyro reference axes. The estimator error equation (7) has an eigenvalue which is a function of the skew angle β and all the gains of the estimator are a function of this one parameter. The simple pole at $-n\tan\beta$ requires $0 \leq \beta \leq \pi/2$ for stability. Within this range the choice of β will depend on the characteristics of the horizon sensor and gyro noise sources. Using Eqs. (7) and (8) the response for the error $\tilde{\varphi}_1(s)$ is

$$\tilde{\varphi}_1(s) = \frac{-\omega_d(s)}{\cos\beta[s+n\tan\beta]} + \frac{v(s)[n-s\tan\beta]}{s+n\tan\beta} \tag{9}$$

If the principal gyro error is a bias term while the horizon sensor contributes high frequency (compared to n) noise which may be modeled as a white source, the gyro drift term contributes ω_d/n at $\beta = \pi/2$ and becomes unbounded at $\beta = 0$, i.e., yaw is not observable by measuring ω_1. Conversely, for high-frequency noise and small β, the estimator attenuates the horizon sensor noise whereas for large β the noise is amplified. Selection of optimal offset angles will be considered below.

Noise Models

Gyro drift is sometimes modeled as a random walk process such that the drift is the first integral of a white noise process; but this grows without bound and is unacceptable for long term behavior which is of interest in gyrocompassing applications.

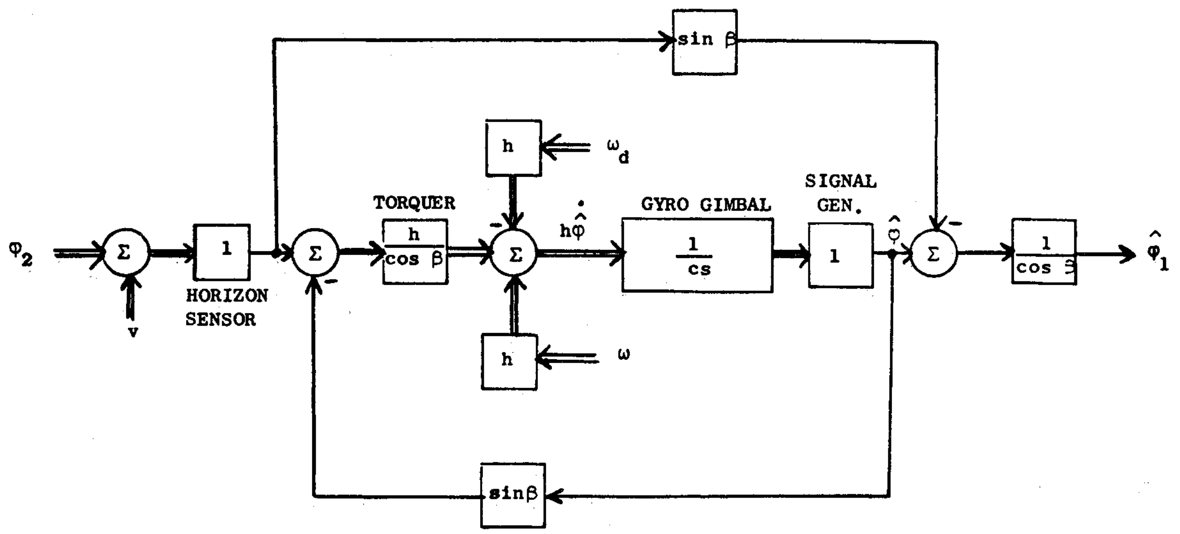

Fig. 2 Reduced state yaw estimator.

A model more acceptable on the basis of gyro behavior is an exponentially correlated process with large correlation time.[3] Such a process may be obtained by passing white noise through a first-order shaping filter[5] so that

$$\dot{\omega}_d' = -\alpha\omega_d' + \alpha w \tag{10}$$

where $\alpha = 1/T_d$, T_d = correlation time, and

$$E[\omega_d'(t)\omega_d'(\tau)] = R_d \exp[-\alpha|t-\tau|]$$

$$Q_d = 2R_d T_d = 2R_d/\alpha, \qquad E[w(t)w(\tau)] = Q_d\delta(t-\tau)$$

Thus, the drift has steady-state mean square value equal to R_d and time correlation exponential with time constant T_d. For times short compared to T_d the model behaves as a random-walk but it eliminates the undesirable long-term feature of the random walk. In addition to the correlated noise, there is usually a random constant drift b such that the total noise is

$$\omega_d(t) = \omega_d'(t) + b \tag{11}$$

Horizon sensor noise results principally from poor horizon definition, especially at low altitudes. Variations in the shape of the atmosphere cloud cover, and seasonal effects can produce errors as large as 10^{-2} rad. Since correlation times for this type of noise are so short compared to orbit period, it may be modeled effectively as white noise, v, such that

$$E[v(t)v(\tau)] = R\delta(t-\tau)$$

The dynamical equations of the estimator error and noise sources may now be written as

$$\begin{bmatrix} \dot{\tilde{\varphi}} \\ \dot{\omega}_d' \\ \dot{b} \end{bmatrix} = \begin{bmatrix} -n\tan\beta & -1 & -1 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\varphi} \\ \omega_d' \\ b \end{bmatrix} + \begin{bmatrix} \dfrac{n}{\cos\beta} & 0 \\ 0 & \alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

$$\tilde{\varphi}_1 = (1/\cos\beta)\tilde{\varphi} - (\tan\beta)v \tag{12}$$

where v and w are white noise sources previously described.

Selection of the skew angle of the gyro input axis may now be made by minimizing a cost functional on the estimator error. A quadratic criterion is selected because it is tractable and reasonable for gyrocompass applications. For the noise sources assumed, the estimator yaw error will consist of a bias plus random components so that

$$\tilde{\varphi}_1 = \tilde{\varphi}_{1b} + \tilde{\varphi}_{1r}(t) \tag{13}$$

and an appropriate unweighted cost functional is

$$J = E(\tilde{\varphi}_{1b}^2) + E\int_0^\infty \tilde{\varphi}_{1r}^2(t)\,dt \tag{14}$$

Since the form of the estimator is specified, the problem is one of parameter optimization. The cost functional can be evaluated in a straightforward manner using Parseval's theorem to transform the time domain integral to the frequency domain where its evaluation is reduced to simple algebraic manipulation. The method is treated in detail in Ref. 6.

Numerical Results

Parseval's theorem, viz.,

$$I = \int_0^\infty x^2(t)\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} x(-s)x(s)\,ds \tag{15}$$

is valid for time functions x having zero value for times less than zero and zero abscissa of convergence. For rational functions $x(s)$ the integral becomes

$$I = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{c(s)c(-s)}{d(s)d(-s)}\,ds \tag{16}$$

where $d(s)$ has zeros only in the left half-plane. Equation (16) is evaluated by solving a set of simultaneous algebraic equations created from Eqs. (12). For a second-order denominator, these equations yield

$$I_2 = \frac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2} \tag{17}$$

The Laplace transform of the estimator yaw error as obtained from Eq. (12) is

$$\tilde{\varphi}_1(s) = \left(\frac{n - s\tan\beta}{s + n\tan\beta}\right)v(s) - \frac{b}{s\cos\beta(s+n\tan\beta)} - \frac{\alpha w(s)}{\cos\beta(s+\alpha)(s+n\tan\beta)} \tag{18}$$

To evaluate the cost functional, the horizon sensor white noise source, v, must be band limited to prevent divergence of the integral in Eq. (14). Using a first-order filter with break frequency at $\gamma = 10n$ provides the desired roll-off characteristics while permitting the noise to appear "white" at orbit frequency. Incorporating the prefilter into the error dynamics, evaluating the integral squared error using Eq. (17), and adding the steady-state bias error yields for the cost functional

$$J = \left(\frac{b}{n\sin\beta}\right)^2 + \frac{\gamma R(\gamma\tan^3\beta + n)}{2\tan\beta(\gamma + n\tan\beta)} + \frac{Q_d\alpha}{2n\cos^2\beta\tan\beta(\alpha + n\tan\beta)} \tag{19}$$

which may be written

$$J(\beta) = Q_b(\beta)b^2 + Q_R(\beta)R + Q_\omega(\beta)Q_d \tag{20}$$

The coefficients Q of the power spectral densities in Eq. (20) are plotted, with appropriate scale factors, in Fig. 3. The noise correlations assumed are $\gamma = 10n$, $\alpha = 0.05n$ (about 30 hr), and $n = 10^{-3}$ rad/sec. The qualitative error responses to white noise disturbances are similar to those obtained in Ref. 2 for single frequency excitation, but there are considerable quantitative differences. Reducing β to zero causes an unbounded estimator error contribution from both correlated and constant gyro errors since the fundamental gyrocompassing information has been lost. For large β, horizon sensor noise creates large errors since the estimator is effectively differentiating the horizon sensor signal to compute yaw.

To use these design curves to select β, a typical nominal set of noise parameters will be chosen and the cost functional plotted to obtain the desired minimum. In problems of this type, it is frequently the case that differentiation to find stationary points leads to transcendental equations in the design parameter. Such is the case here and plotting obviates the need for tedious

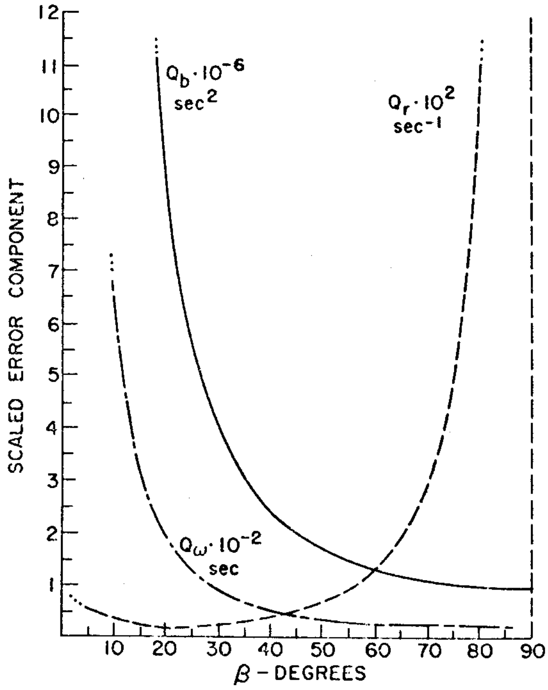

Fig. 3 Contributions to total squared error.

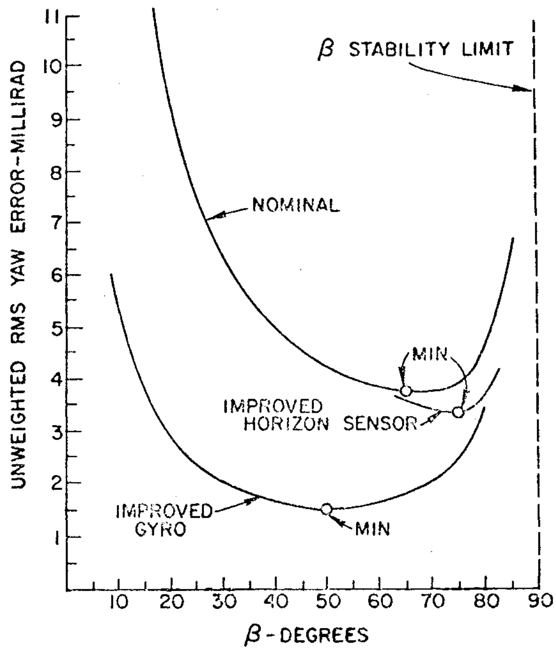

Fig. 4 Rms estimator error vs skew angle.

algebraic manipulations. The nominal performance of the sensors is given in the table below.

Table 1 Nominal measurement errors

Error	Magnitude	Power spect. dens.
Gyro bias:	$(\frac{2}{3})^\circ$/hr	10^{-11} rad^2/sec^2
Gyro correlated:	0.2°/hr	10^{-12} rad^2/sec
HS correlated:	0.5°	10^{-4} rad^2/sec

These values are typical of current, commercially available, flight proven satellite hardware. With these power spectral densities it is seen from Fig. 3 that the gyro bias and horizon sensor noise terms contribute comparable errors while the correlated gyro noise component contributes little to the total noise error. This phenomena is the basis for the successful operation of the gyrocompass, i.e., the gyro provides fast response capability with the low-frequency gyro drift controlled by the horizon sensor. Large gyro noise jitter effects would negate the value of this instrument combination. The square root of the total squared error is plotted in Fig. 4 for the nominal case and two additional "high performance" cases. For the nominal instrument, the rms error is about 0.25° at $\beta = 65°$. If the horizon sensor noise power spectral density is reduced one order of magnitude (currently achievable at much higher cost), only a small improvement in performance is noted at $\beta = 75°$. With the nominal horizon sensor and the same improvement in gyro bias error, there is a marked improvement in performance. The over-all rms error is reduced to approximately 0.1° and the error sensitivity to skew angle variations (about $\beta = 50°$) is significantly reduced.

The noise filtering characteristics exhibited in Figs. 3 and 4 are a consequence of estimator bandwidth adjustment through the skew angle parameter. Large skew angles permit the gyrocompass to quickly recover from transients and initial conditions at the expense of poor long term performance. For typical applications, e.g., gravity-stabilized satellites, long term performance is far more important than transient response and, consequently, practical sensor improvements should be concentrated in the gyro rather than in the horizon sensor. These remarks are based on the unweighted rms errors given in Fig. 4. For special applications, additional weighting coefficients might be appropriate in Eq. (20).

Conclusions

The development of the skewed single gyro gyrocompass previously obtained[2] may be replaced with a derivation, based on the Luenberger reduced state observer theory,[4] which automatically specifies all feedforward and feedback gains. Stochastic noise models for gyro and horizon sensor instruments show that over-all yaw errors of 0.1° are obtainable below synchronous altitude with commercially available gyros. For systems where difficult sensor noise environments are not a problem, the reduced state yaw estimator can provide reliable and accurate measurement without the cost and complexity of star reference or 2 gyro gyrocompassing systems. Improvements in the long term performance of the instrument may be most efficiently made by concentrating the principal improvement in the gyro rather than in the horizon sensor for the case evaluated.

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